The term Hermitian is used to describe the fact that $(\cdot,\cdot)$ is not symmetric, but rather satisfies the relation

$$(z,w) = \overline{(w,z)} \quad \text{for all } z,w \in \mathbb{C}.$$

Show that

$$\langle z,w\rangle = \frac{1}{2}[(z,w)+(w,z)] = \mathrm{Re}(z,w),$$

where we use the usual identification $z = x+iy \in \mathbb{C}$ with $(x,y)\in\mathbb{R}^2$.

3. With $\omega = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where n is a natural number. How many solutions are there?

4. Show that it is impossible to define a total ordering on $\mathbb{C}$. In other words, one cannot find a relation $\succ$ between complex numbers so that:

(i) For any two complex numbers z, w, one and only one of the following is true: $z \succ w$, $w \succ z$ or $z = w$.

(ii) For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \succ z_2$ implies $z_1 + z_3 \succ z_2 + z_3$.

(iii) Moreover, for all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \succ 0$, then $z_1 \succ z_2$ implies $z_1 z_3 \succ z_2 z_3$.

[Hint: First check if $i \succ 0$ is possible.]

5. A set Ω is said to be **pathwise connected** if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an *open* set Ω is pathwise connected if and only if Ω is connected.

(a) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parametrization $z : [0,1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

$$t^* = \sup_{0\leq t\leq 1} \{t : z(s) \in \Omega_1 \quad \text{for all } 0 \leq s < t\}.$$

Arrive at a contradiction by considering the point $z(t^*)$.

(b) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega_1$ as desired.

The proof actually shows that the regularity and type of curves we used to define pathwise connectedness can be relaxed without changing the equivalence between the two definitions when Ω is open. For instance, we may take all curves to be continuous, or simply polygonal lines.[2]

6. Let Ω be an open set in $\mathbb{C}$ and $z \in \Omega$. The **connected component** (or simply the **component**) of Ω containing z is the set $\mathcal{C}_z$ of all points w in Ω that can be joined to z by a curve entirely contained in Ω.

(a) Check first that $\mathcal{C}_z$ is open and connected. Then, show that $w \in \mathcal{C}_z$ defines an equivalence relation, that is: (i) $z \in \mathcal{C}_z$, (ii) $w \in \mathcal{C}_z$ implies $z \in \mathcal{C}_w$, and (iii) if $w \in \mathcal{C}_z$ and $z \in \mathcal{C}_\zeta$, then $w \in \mathcal{C}_\zeta$.

Thus Ω is the union of all its connected components, and two components are either disjoint or coincide.

(b) Show that Ω can have only countably many distinct connected components.

(c) Prove that if Ω is the complement of a compact set, then Ω has only one unbounded component.

[Hint: For (b), one would otherwise obtain an uncountable number of disjoint open balls. Now, each ball contains a point with rational coordinates. For (c), note that the complement of a large disc containing the compact set is connected.]

7. The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

(a) Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \qquad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1 \qquad \text{if } |z| = 1 \text{ or } |w| = 1.$$

[Hint: Why can one assume that z is real? It then suffices to prove that

$$(r-w)(r-\overline{w}) \leq (1-rw)(1-r\overline{w})$$

with equality for appropriate r and $|w|$.]

(b) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

[2] A polygonal line is a piecewise-smooth curve which consists of finitely many straight line segments.